

NOTE ON THE CONJECTURE OF D. BLAIR IN CONTACT RIEMANNIAN GEOMETRY.

VLADIMIR KROUGLOV.

ABSTRACT. The conjecture of D. Blair says that there are no nonflat Riemannian metrics of nonpositive curvature compatible with a contact structure. We prove this conjecture for a certain class of contact structures on closed 3-dimensional manifolds and construct a local counterexample. We also prove that a hyperbolic metric on \mathbb{R}^3 cannot be compatible with any contact structure.

1. INTRODUCTION

In [2, p. 99] author states the following conjecture:

Conjecture 1.1. *There are no nonflat Riemannian metrics of nonpositive curvature that are compatible with a contact structure.*

On a 3-torus, standard Euclidean metric is compatible with a contact structure given by the kernel of the one-form $\cos(z)dx + \sin(z)dy$. Despite the fact that there exist contact structures on higher dimensional tori [3], as it was shown in [1] flat metric cannot be compatible with any contact structure when the dimension of a manifold is greater than 3.

Using the result of A. Zeghib [10] on the existence of geodesic flows, on closed manifolds the conjecture of D. Blair is true for the Riemannian metrics of strictly negative sectional curvature. In [6], it has been shown that the conjecture is true for the homogenous Riemannian metric adapted to a homogenous contact structure.

Note, that in view of the results in [5] closed contact metric manifolds of nonpositive curvature would provide a source of examples of tight contact structures.

The main result of the present paper is the proof of the D. Blair's conjecture for contact structures which are sufficiently nontrivial as fibrations. We prove the following

Theorem 1.2. *Assume that M is a closed 3-manifold with a contact structure ξ which cannot be decomposed as a sum of two one-dimensional fibrations $\xi \neq \eta_1 \oplus \eta_2$. Then the conjecture of D. Blair is true for (M, ξ) .*

By the result of Z. Olszak in [7], when the dimension of a manifold is greater than three constant negative curvature metrics cannot be compatible with a contact structure (even when the manifold is not compact). Analyzing the curvature tensor of a compatible metric we prove this result in dimension three.

Proposition 1.3. *Constant negative curvature metric on a 3-manifold cannot be compatible with a contact structure.*

We end with a local counterexample to the conjecture of D. Blair. We construct a Riemannian metric compatible with a standard contact structure on \mathbb{R}^3 which has strictly negative curvature in some neighborhood of zero in \mathbb{R}^3 .

2. CONTACT METRIC MANIFOLDS.

2.1. Compatible metrics. Assume that (M, ξ) is a contact 3-manifold. If we fix a one-form α among the conformal class $\{f\alpha' : \text{for positive functions } f \text{ on } M\}$ which we call the contact one-form associated with the contact structure then there is a unique vector field N called the Reeb vector field of α such that

$$\alpha(N) = 1, \quad L_N \alpha = \iota_N d\alpha = 0$$

Let J be an almost complex structure on ξ (i.e. $J^2 = -id$). We may complement it to a linear operator on TM by setting $JN = 0$.

Definition 2.1. A Riemannian metric $\langle \cdot, \cdot \rangle$ is called compatible with ξ if there is an associated 1-form α and an almost complex structure J such that

$$\langle N, X \rangle = \alpha(X), \quad k\langle X, JY \rangle = d\alpha(X, Y)$$

where k is some constant and X and Y are the vector fields on M .

By a contact metric manifold we are going to understand the tuple $(M, \xi, \alpha, \langle \cdot, \cdot \rangle, J)$.

2.2. Second fundamental form. The second fundamental form of a plane field is a symmetric bilinear form which generalizes the corresponding notion for a surface inside the Riemannian manifold. The following definition is due to Reinhart [8]

Definition 2.2. The second fundamental form of plane field ξ is a bilinear form on ξ defined as

$$II(X, Y) = \frac{1}{2} \langle \nabla_X Y + \nabla_Y X, N \rangle$$

where X and Y are in ξ , N is a unit normal vector field to ξ and ∇ is a Levi-Civita connection of $\langle \cdot, \cdot \rangle$.

We are going to call the linear operator A_N which corresponds to II with respect to $\langle \cdot, \cdot \rangle$ – a *shape operator* of ξ . Since II is symmetric, the shape operator has two real eigenvalues that we call the principal curvatures of ξ . The eigenvectors of A_N will be called the principal directions of ξ . We also define the *extrinsic curvature* K_e and the *mean curvature* H of ξ as the determinant and the half trace of the *shape operator* correspondingly. When the plane field ξ is integrable, the second fundamental form of ξ coincides with a second fundamental forms of the integral surfaces. All notions of the classic surface theory extend naturally to the context of plane distributions.

2.3. Extrinsic geometry in compatible metric. When M is a contact metric manifold, the contact structure ξ has a very special geometry with respect to the compatible metric $\langle \cdot, \cdot \rangle$. We have the following

Proposition 2.3. [2] *With respect to a compatible metric, the Reeb vector field is a unit speed geodesic vector field and the contact structure is minimal.*

We are also going to summarize several properties of the contact structures with respect to a compatible metric that will be used in the derivation of the curvature tensor.

Lemma 2.4. *Let $(M, \xi, \alpha, \langle \cdot, \cdot \rangle, J)$ be a contact metric manifold. Then,*

- (1) *J is a rotation by $\frac{\pi}{2}$ in ξ .*
- (2) *For every pair of orthonormal vectors X and Y in ξ the function $\langle [X, Y], N \rangle = \pm k$.*

(3) If X and Y are unit orthogonal principal directions of A_N then

$$\langle \nabla_X Y, N \rangle = -\langle \nabla_Y X, N \rangle = \pm \frac{k}{2}.$$

Proof: For every pair of vectors X and Y in ξ

$$k\langle JX, JY \rangle = d\alpha(JX, Y) = -d\alpha(Y, JX) = -k\langle Y, J^2 Y \rangle = k\langle X, Y \rangle.$$

We are left to check that X is orthogonal to JX . This follows from

$$k\langle X, JX \rangle = d\alpha(X, X) = 0$$

If X and Y are orthonormal, then $Y = \pm JX$. We have

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) = -\langle [X, Y], N \rangle.$$

On the other hand

$$d\alpha(X, Y) = \pm k\langle X, X \rangle = \pm k$$

which proves (2).

Since X and Y are the eigenvectors of A_N , $\frac{1}{2}\langle \nabla_X Y + \nabla_Y X, N \rangle = 0$. From (2),

$$\langle \nabla_X Y, N \rangle = \frac{1}{2}\langle \nabla_X Y + \nabla_Y X, N \rangle + \frac{1}{2}\langle \nabla_X Y - \nabla_Y X, N \rangle = \pm \frac{k}{2}$$

3. CURVATURE TENSOR OF THE COMPATIBLE METRIC ON A 3-MANIFOLD.

In this section we are going to compute the matrix of the curvature tensor of a compatible metric. Assume that $(M, \xi, \alpha, \langle \cdot, \cdot \rangle, J)$ is a contact metric manifold. Let N be the Reeb vector field of α . Denote by X and Y the (local) orthonormal frame in ξ that consists of the eigenvectors of the shape operator at a given point $p \in M$.

Let λ be a principal curvature that corresponds to a principal direction X . Since ξ is minimal, the mean curvature of ξ vanishes and Y corresponds to the principal curvature $-\lambda$.

Lemma 3.1. *With respect to a basis of bivectors $X \wedge Y$, $X \wedge N$ and $Y \wedge N$ the matrix of the curvature tensor of $\langle \cdot, \cdot \rangle$ is given by*

$$\mathcal{R} = \begin{pmatrix} -\frac{3k^2}{4} + \lambda^2 + K & -Y(\lambda) - 2\lambda\langle \nabla_X X, Y \rangle & X(\lambda) - 2\lambda\langle \nabla_Y Y, X \rangle \\ -Y(\lambda) - 2\lambda\langle \nabla_X X, Y \rangle & \frac{k^2}{4} - \lambda^2 + N(\lambda) & 2\lambda\langle \nabla_N X, Y \rangle \\ X(\lambda) - 2\lambda\langle \nabla_Y Y, X \rangle & 2\lambda\langle \nabla_N X, Y \rangle & \frac{k^2}{4} - \lambda^2 + N(\lambda) \end{pmatrix}$$

where

$$K = X(\langle \nabla_Y Y, X \rangle) + Y(\langle \nabla_X X, Y \rangle) - \langle \nabla_Y Y, X \rangle^2 - \langle \nabla_X X, Y \rangle^2 - \langle [X, Y], N \rangle \langle [N, Y], X \rangle$$

is the curvature of a generalized Webster connection (see [9] for the definition) and λ is an eigenvalue of the shape operator which corresponds to X .

Proof: By replacing X by $-X$ if required we may assume that $\langle [X, Y], N \rangle = k$.

Calculation of $\mathcal{R}_{11} = \langle R(X, Y)Y, X \rangle$.

$$\langle R(X, Y)Y, X \rangle = \langle \nabla_X \nabla_Y Y, X \rangle - \langle \nabla_Y \nabla_X Y, X \rangle - \langle \nabla_{[X, Y]} Y, X \rangle$$

The first summand is

$$\begin{aligned} \langle \nabla_X \nabla_Y Y, X \rangle &= X(\langle \nabla_Y Y, X \rangle) - \langle \nabla_Y Y, \nabla_X X \rangle \\ &= X(\langle \nabla_Y Y, X \rangle) - \langle \nabla_Y Y, N \rangle \langle \nabla_X X, N \rangle = X(\langle \nabla_Y Y, X \rangle) + \lambda^2 \end{aligned}$$

The second summand is

$$-\langle \nabla_Y \nabla_X Y, X \rangle = -Y(\langle \nabla_X Y, X \rangle) + \langle \nabla_X Y, \nabla_Y X \rangle = Y(\langle \nabla_X X, Y \rangle) - \frac{k^2}{4}$$

as follows from (3) in Lemma 2.4. The third summand is

$$\begin{aligned} -\langle \nabla_{[X,Y]} Y, X \rangle &= -\langle [X, Y], X \rangle \langle \nabla_X Y, X \rangle - \langle [X, Y], Y \rangle \langle \nabla_Y Y, X \rangle - \langle [X, Y], N \rangle \langle \nabla_N Y, X \rangle \\ &= -\langle \nabla_X X, Y \rangle^2 - \langle \nabla_Y Y, X \rangle^2 - \langle [X, Y], N \rangle (\langle \nabla_Y N, X \rangle + \langle [N, Y], X \rangle) \\ &= -\langle \nabla_X X, Y \rangle^2 - \langle \nabla_Y Y, X \rangle^2 - \frac{k^2}{2} - \langle [X, Y], N \rangle \langle [N, Y], X \rangle \end{aligned}$$

Summing this up will give us the desired expression for \mathcal{R}_1 .

Calculation of $\mathcal{R}_{22} = \langle R(X, N)N, X \rangle$.

$$\langle R(N, X)X, N \rangle = \langle \nabla_N \nabla_X X, N \rangle - \langle \nabla_X \nabla_N X, N \rangle - \langle \nabla_{[N,X]} X, N \rangle$$

The first summand is

$$\langle \nabla_N \nabla_X X, N \rangle = \nabla_N \langle \nabla_X X, N \rangle - \langle \nabla_X X, \nabla_N N \rangle = N(\lambda)$$

The second summand is

$$\begin{aligned} -\langle \nabla_X \nabla_N X, N \rangle &= -X \langle \nabla_N X, N \rangle + \langle \nabla_N X, \nabla_X N \rangle \\ &= -X(N \langle X, N \rangle - \langle X, \nabla_N N \rangle) + \langle \nabla_N X, \nabla_X N \rangle = \langle \nabla_N X, \nabla_X N \rangle. \end{aligned}$$

Here we used that N is a geodesic vector field. Finally, the last summand is

$$\begin{aligned} -\langle \nabla_{[N,X]} X, N \rangle &= -\langle [N, X], X \rangle \langle \nabla_X X, N \rangle + -\langle [N, X], Y \rangle \langle \nabla_Y X, N \rangle \\ &= -\lambda^2 - \langle [N, X], Y \rangle \langle \nabla_Y X, N \rangle \end{aligned}$$

Summing these expressions we get

$$K(N, X) = N(\lambda) - \lambda^2 - \langle [N, X], Y \rangle \langle \nabla_Y X, N \rangle + \langle \nabla_N X, Y \rangle \langle Y, \nabla_X N \rangle$$

Using (2) and (3) of Lemma 2.4 we get

$$K(N, X) = N(\lambda) - \lambda^2 - \frac{k}{2} (\langle [N, X], Y \rangle - \langle \nabla_N X, Y \rangle) = N(\lambda) - \lambda^2 + \frac{k^2}{4}$$

Calculation of $\mathcal{R}_{33} = \langle R(Y, N)N, Y \rangle$.

By exactly the same calculations replacing X by Y we get

$$\langle R(Y, N)N, Y \rangle = -N(\lambda) - \lambda^2 + \frac{k^2}{4}.$$

Calculation of $\mathcal{R}_{23} = \langle R(X, N)N, Y \rangle$.

$$\langle R(X, N)N, Y \rangle = \langle \nabla_X \nabla_N N, Y \rangle - \langle \nabla_N \nabla_X N, Y \rangle - \langle \nabla_{[X,N]} N, Y \rangle$$

Obviously, since N is geodesic the first summand is zero. Rewrite the second summand,

$$\begin{aligned} -\langle \nabla_N \nabla_X N, Y \rangle &= -N(\langle \nabla_X N, Y \rangle) + \langle \nabla_X N, \nabla_N Y \rangle = \langle \nabla_X N, \nabla_N Y \rangle \\ &= \langle \nabla_X N, X \rangle \langle X, \nabla_N Y \rangle + \langle \nabla_X N, Y \rangle \langle Y, \nabla_N Y \rangle = -\lambda \langle X, \nabla_N Y \rangle \end{aligned}$$

The last summand

$$-\langle \nabla_{[X,N]} N, Y \rangle = -\langle [X, N], X \rangle \langle \nabla_X N, Y \rangle - \langle [X, N], Y \rangle \langle \nabla_Y N, Y \rangle$$

Summing these expressions we get:

$$\langle R(Y, N)N, Y \rangle = \lambda \langle \nabla_N X, Y \rangle + \lambda \langle \nabla_X N, Y \rangle - \lambda \langle [X, N], Y \rangle = 2\lambda \langle \nabla_N X, Y \rangle$$

Calculation of $\mathcal{R}_{13} = \langle R(X, Y)N, Y \rangle$.

$$\langle R(X, Y)N, Y \rangle = -\langle R(X, Y)Y, N \rangle = -\langle \nabla_X \nabla_Y Y, N \rangle + \langle \nabla_Y \nabla_X Y, N \rangle + \langle \nabla_{[X, Y]} Y, N \rangle$$

The first summand is

$$\begin{aligned} -\langle \nabla_X \nabla_Y Y, N \rangle &= -X \langle \nabla_Y Y, N \rangle + \langle \nabla_Y Y, \nabla_X N \rangle = X(\lambda) + \langle \nabla_Y Y, X \rangle \langle X, \nabla_X N \rangle \\ &= X(\lambda) - \lambda \langle \nabla_Y Y, X \rangle \end{aligned}$$

The second summand is

$$\begin{aligned} \langle \nabla_Y \nabla_X Y, N \rangle &= Y(\langle \nabla_X Y, N \rangle) - \langle \nabla_X Y, \nabla_Y N \rangle = -\langle \nabla_X Y, X \rangle \langle X, \nabla_Y N \rangle \\ &= -\langle \nabla_X X, Y \rangle \langle N, \nabla_Y X \rangle \end{aligned}$$

Finally, the last summand,

$$\begin{aligned} \langle \nabla_{[X, Y]} Y, N \rangle &= \langle [X, Y], X \rangle \langle \nabla_X Y, N \rangle + \langle [X, Y], Y \rangle \langle \nabla_Y Y, N \rangle \\ &= -\langle \nabla_X X, Y \rangle \langle N, \nabla_X Y \rangle - \lambda \langle \nabla_Y Y, X \rangle \end{aligned}$$

Summing this up gives us

$$\langle R(X, Y)N, Y \rangle = X(\lambda) - 2\lambda \langle \nabla_Y Y, X \rangle$$

Calculation of $\mathcal{R}_{12} = \langle R(X, Y)N, X \rangle$.

Analogously,

$$\langle R(X, Y)N, X \rangle = -Y(\lambda) - 2\lambda \langle \nabla_X X, Y \rangle$$

Corollary 3.2. *Assume that M is a closed 3-manifold with a contact structure ξ which cannot be decomposed as a sum of two one-dimensional fibrations $\xi \neq \eta_1 \oplus \eta_2$. Then the conjecture of D. Blair is true for (M, ξ) .*

Proof: Under the assumptions of the corollary, for every Riemannian metric g on M , ξ must have an *umbilic point*. At this point we have $\lambda = 0$ and

$$K(X, N) + K(Y, N) = \frac{k^2}{2} - 2\lambda^2 = \frac{k^2}{2} > 0$$

Therefore, g cannot have nonpositive curvature.

Proposition 3.3. *Constant negative curvature metric cannot be compatible with any contact structure.*

Proof: Since the sectional curvature of the metric is constant, it is easy to see that the principal curvatures should also be constant on M . With respect to a basis of bivectors $X \wedge Y$, $X \wedge N$ and $Y \wedge N$ the matrix of the curvature operator of $\langle \cdot, \cdot \rangle$ is diagonal. Noting that λ cannot be zero,

$$\begin{cases} \langle \nabla_N X, Y \rangle = 0 \\ \langle \nabla_X X, Y \rangle = 0 \\ \langle \nabla_Y Y, X \rangle = 0 \end{cases}$$

The Webster curvature with respect to $\langle \cdot, \cdot \rangle$ is

$$\begin{aligned} K &= X(\langle \nabla_Y Y, X \rangle) + Y(\langle \nabla_X X, Y \rangle) - \langle \nabla_Y Y, X \rangle^2 \\ &\quad - \langle \nabla_X X, Y \rangle^2 - \langle [X, Y], N \rangle \langle [N, Y], X \rangle \\ &= -\frac{k}{2}(\langle \nabla_N Y, X \rangle - \langle \nabla_Y N, X \rangle) = \frac{k}{2}(\langle \nabla_Y N, X \rangle) = \frac{k^2}{2} \end{aligned}$$

Since $K(X, Y) = K(X, N)$ we have that

$$-\frac{3k^2}{4} + \frac{k^2}{2} + \lambda^2 = \frac{k^2}{4} - \lambda^2,$$

which implies that $\lambda = \frac{1}{2}$ and the metric is flat.

4. LOCAL COUNTEREXAMPLE TO THE CONJECTURE OF D. BLAIR.

On \mathbb{R}^3 with cartesian coordinates (x, y, z) consider a standard contact structure ξ given by the kernel of the one-form $\alpha = dz - xdy$. We will construct a Riemannian metric which would be compatible with ξ and have nonpositive (even strictly negative) curvature in some neighborhood of zero in \mathbb{R}^3 .

With respect to this metric the Reeb vector field of α has to be a unit geodesic vector field and ξ has to be a minimal distribution. It is easy to check that in this case the matrix of g should have the form

$$g = \begin{pmatrix} a & b & 0 \\ b & c & x \\ 0 & x & 1 \end{pmatrix}.$$

where the functions a, b and c additionally satisfy the condition

$$H = \frac{1}{2} \frac{\partial}{\partial z} (a(c - x^2) - b^2) = 0$$

This condition will be automatically satisfied if we choose

$$\begin{cases} a = Ae^z \\ b = 1 \\ c = x^2 + Be^{-z} \end{cases}$$

With respect to an orthonormal frame $(\frac{\partial}{\partial z}, \frac{1}{\sqrt{Ae^z}} \frac{\partial}{\partial x}, \sqrt{\frac{Ae^z}{AB-1}} (-\frac{1}{Ae^z} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}))$ the curvature tensor is given by a matrix

$$\begin{pmatrix} \frac{1}{4} \frac{AB-3-2x^2Ae^z}{AB-1} & -\frac{1}{2}x\sqrt{\frac{Ae^z}{AB-1}} & \frac{1}{2} \frac{x\sqrt{Ae^z}}{AB-1} \\ -\frac{1}{2}x\sqrt{\frac{Ae^z}{AB-1}} & -\frac{1}{4} & 0 \\ \frac{1}{2} \frac{x\sqrt{Ae^z}}{AB-1} & 0 & \frac{1}{4} \end{pmatrix}$$

Clearly when $AB \in (1, 3)$, the matrix of the curvature tensor is negatively definite in some neighborhood of zero in \mathbb{R}^3 .

REFERENCES

- [1] *D. E. Blair.*, On the non-existence of flat contact metric structures, *Tohoku Math. J.*, 28, 373-379. 1976
- [2] *D. E. Blair.* Riemannian geometry of contact and symplectic manifolds. *Progress in Mathematics*, 2002
- [3] *F. Bourgeois*, Odd dimensional tori are contact manifolds, *International Mathematics Research Notices*, no. 30, 1571-1574
- [4] *S.S. Chern and R.S. Hamilton.* On Riemannian metrics adapted to three-dimensional contact manifolds. In *Workshop Bonn 1984 (Bonn, 1984)*, vol. 1111 of *Lecture Notes in Math.*, pages 279-308. Springer, Berlin, 1985
- [5] *J.B. Etnyre, R. Komendarczyk, P. Massot*, Tightness in contact metric manifolds, preprint
- [6] *A. Lotta* Non existence of homogenous contact metric manifolds of nonpositive curvature, *Tohoku Math. J.*, 62 (2010), 575-578
- [7] *Z. Olszak*, On contact metric manifolds, *Tohoku Math. J.*, 31, 247-253, 1979.

- [8] *B.Reinhart*. The second fundamental form of a plane field. - Cambridge University Press, *J. Diff. Geom.* 1977. - p 619–627.
- [9] *S. Tanno*, Variational problems on contact Riemannian manifolds, Transactions of the AMS, v.314, 1, 1989, p. 349–379
- [10] *A. Zeghib*. Sur les feuilletages geodesiques continus des variétés hyperboliques. Invent. Math. 114(1):193-206, 1993